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Decoherence and quantum error correction

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We discuss the effects of decoherence on fault-tolerant quantum computation and how it leads to accuracy thresholds required before arbitrarily computations can be performed. We examine the feasibility of achieving these thresholds in an ion trap realization when spontaneous emission is taken into account.

1. Introduction

Since Shor's discovery (Shor 1994; Ekert *et al.* 1996) of an algorithm that allows the factorization of a large number by a quantum computer in polynomial time instead of an exponential time as in classical computing, interest in the practical realization of a quantum computer has been much enhanced. Recent advances in the preparation and manipulation of single ions as well as the engineering of pre-selected cavity light fields suggests that quantum optics may well be that field of physics promising the first experimental realization of a quantum computer.

The realization of a quantum computer in a linear ion trap seemed very promising as it was thought that decoherence could be suppressed sufficiently to preserve the superpositions necessary for quantum computation. Indeed, a single quantum gate in such an ion trap has been realized by Monroe *et al.* (1995). Nevertheless, the error rate in this experiment from technical sources was too high to allow the realization of extended quantum networks. However, there remains the question whether overcoming technical problems will be sufficient to realize practically useful computations. Here we address the problem of so called threshold accuracy in quantum computation (Knill *et al.* 1996; Aharonov & Ben-Or 1996). Arbitrarily complicated (long) quantum computations can be performed once the error rate of a quantum gate can be pushed below a certain threshold. We will discuss whether the required thresholds presented in Knill *et al.* (1996) and Aharonov & Ben-Or (1996) can be achieved or if spontaneous emission rules this out.

2. Accuracy thresholds to quantum computation

An input to a quantum computer is a string of quantum bits called *qubits*. A quantum computer is viewed as consisting of two main parts: *quantum gates* and *quantum wires*. By *basic* quantum gates we mean any set of quantum gates which can perform any desired quantum computation. A universal quantum gate is the one whose combination can be used to simulate any other quantum gate. A quantum wire is used as a representation of that part of computation of any qubit where the evolution is a simple identity operation (i.e. no gate operates on the qubit), as well as the time the qubit spends during the gate operation.

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There are two main types of errors occurring during a quantum computation process. The first type are decoherence and spontaneous emission errors due to the interaction of qubits with the external environment. These errors are represented as unitary evolutions of the joint qubit–environment system after which the environment is traced out. This transforms an initially pure state into a statistical mixture of qubits. The most general interaction of this kind can be represented by positive operator valued measure (POVM) type operators acting on the qubit only,

$$\rho \to \sum A_i \rho A_i^{\dagger}, \tag{2.1}$$

where $\sum A_i^{\dagger} A_i = I$. This is an incoherent type of error.

The second type of error is an error in quantum gates where the qubit is 'over rotated' by a certain small amount. This is a coherent type of error which is simply represented by a unitary transformation on the qubit $\rho \to V_i \rho V_i^{\dagger}$, where the 'true' transformation should have been $\rho \to U_i \rho U_i^{\dagger}$. The difference between U and V signifies the size of the error. This type of error can be represented as a special case of POVM operators in equation (2.1), but the error it produces is different in nature to incoherent error.

3. Fault-tolerant computation

The idea of fault tolerant quantum computation (Shor 1994, 1995) is to encode the qubits in such a way that the encoding does not introduce more errors than previously were present. If the error stays at the same level, we then keep performing error correction until the error has decreased in magnitude (Shor 1995; DiVincenzo & Shor 1996; Plenio *et al.* 1997). The present state of the art requires 5–10 qubits to encode a single qubit against a single error. It is the iterative application 'in depth' of the encoding that will enable us to reduce error to an arbitrarily small level providing it is below a certain level to start with. In other words, we will be encoding the encoding bits. We list the assumptions that we use below.

1. Qubits errors occur independently.

2. There are three basic incoherent errors which occur at the same rate (isotropic decay is assumed for simplicity). In the Born-Markov approximation, the master equation for a single qubit is of the Lindblad form. Three basic single qubit errors are represented by the Pauli spin matrices. We assume that the probability that there is no error in n qubits after time t is

$$p_{\mathrm{ne}} = \mathrm{e}^{-n\Gamma t} \tag{3.1}$$

where $\Gamma = 3\gamma$ is the decay rate for all the errors together. The probability that there is *at least one error* is therefore $p_{\mathrm{e}} = 1 - p_{\mathrm{ne}}$. We define $\eta = \Gamma t$ henceforth.

3. Incoherent and coherent errors are independent.

4. Errors are treated 'classically'. We assume that the coherent error per gate is ϵ (no error is $1 - \epsilon$) in a completely classical fashion; this implies that either the qubit was correctly rotated with probability $1 - \epsilon$ or it was over-rotated with the probability ϵ.

5. Basic gates are all two-qubit gates. It is true that almost all two-bit gates are universal. Shor (1994), however, also used a Toffoli (three-bit) gate for his fault-tolerant computation, but this does not change our argument.

6. The number of gates needed to encode and fault tolerantly correct a basic gate are of the order $\mathcal{O}(l^2)$, where l is the overall number of qubits (Shor 1996).

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For stable quantum computation, obviously, we require that the probability of error after the fault-tolerantly encoded basic gate is of higher order (i.e. the error is smaller) than the probability of error after the unencoded gate (that is the whole point of encoding and fault-tolerant error correction!). From this we derive the bound on the size of allowed errors in the wires and in the gates. When we encode the encoding bits again, we reduce the error further and can reduce the error arbitrarily for an arbitrarily long computation. Therefore, given certain initial limits on the error rate in the gates and wires we can stabilize any computation to a desirably small error rate, given an unlimited amount of time. The probability of having any of the three basic errors in the first as well as in the second wire is η, giving the overall first-order wire error of 2η. The error in the gate is ϵ. We assume that the overall error of the whole basic gate is less than or equal to $2\eta+\epsilon$. Suppose that the basic gate is now encoded fault tolerantly against a single error of any kind, using l qubits. Then the overall second-order error is at the end of the gate,

$$\eta^*(\eta,\epsilon,l) = (1-\tfrac{1}{2}l(l-1)l^4\eta^2)l^2\epsilon + \tfrac{1}{2}l(l-1)l^4\eta^2(1-l^2\epsilon), \tag{3.2}$$

i.e. equal to having error in the wires (this time in second order) and not in the gates plus having error in the gates and not in the wires. The term $\frac{1}{2}l(l-1)$ comes from choosing two out of l gates to err and the factor l^4 derives from the use of l^2 gates, so that the error is transformed according to $\eta \to l^2\eta$ and is of second order. We require that the fault tolerant error correction reduces the error. Hence

$$(1-\tfrac{1}{2}l(l-1)l^4\eta^2)l^2\epsilon + \tfrac{1}{2}l(l-1)l^4\eta^2(1-\epsilon) \leqslant 2\eta+\epsilon. \tag{3.3}$$

As the left-hand side is greater than η, we simplify the above without a greater loss in generality to

$$(1-\tfrac{1}{2}l(l-1)l^4\eta^2)l^2\epsilon + \tfrac{1}{2}l(l-1)l^4\eta^2(1-\epsilon) \leqslant \eta. \tag{3.4}$$

The solutions to the equation derived from the above are

$$\eta_\pm = \frac{1 \pm \sqrt{1-2(l^8\epsilon - 2l^{10}\epsilon^2)}}{(l^6-2l^8\epsilon)}. \tag{3.5}$$

We require that $\eta \in \mathcal{R}$ (and that $0 \leqslant \eta \leqslant \frac{1}{2}$) so that we have the following two regimes of error: (1) $0<\eta<\eta_+$ and $e \leqslant e_-$ and (2) $0<\eta<\eta_-$ and $e \geqslant e_+$, where $\epsilon_\pm = (1/2l^2)(1 \pm \sqrt{1-2l^{-6}})$.

The output of the first encoded basic gate is fed into the next one (or part of the output into one next basic gate and the rest into another next basic gate). It is evident that if condition 1 holds, further encoding can only decrease the error. The residual error not taken into account is *ca.* $l^3(l^2\eta)^3 = l^9\eta^3$ (i.e. the second-order error is not corrected by our encoding). In the worst case when $\epsilon=\epsilon_- \sim l^{-8}$, we get $\eta \sim l^{-6}$, which means that the residual uncorrected error is *ca.* l^{-9}. This error can accumulate over time if the computation is sufficiently long. However, the residual error after n in depth encodings is $l^{-\mathcal{O}(n)}$, which can made be arbitrarily small using sufficiently large n. Unfortunately, in this case, the number of elementary gates needed for each new encoding would square, i.e. the time of computation (*ca.* the number of elementary gates) would increase exponentially. Whereas in theory any computation can be performed with arbitrary accuracy given an initial $\eta \sim l^{-6}$ per gate, the time of computation will grow as 1/(residual error). In the second case, however, $\eta_+ \leqslant 0$ and $\eta_- \geqslant \frac{1}{2}$. Thus in the best case, when $\epsilon=\epsilon_+$, η tends asymptotically to $\frac{1}{2}$. This would imply that a qubit initially pure state, e.g. $|0\rangle$, would eventually evolve into a completely random mixture at the output, of the form $\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. Therefore under condition 2, the error rate is too big to allow any useful computation. However, under condition 1, the stabilization of error is clearly accomplished.

4. The error rate in one quantum gate

We have seen how to estimate the accuracy threshold for quantum computation and we have given the numbers that arise from more precise explicit constructions of error correction schemes. We have seen that the incoherent error rate per quantum gate should not be higher than around 10^{-6}. In a more detailed analysis (Knill *et al.* 1996), the execution of one quantum gate on an encoded qubit requires of the order of $N = 10^6$ operations, which confirms our qualitative arguments. We now see whether accuracies of that order can be achieved in a linear ion trap realization of the quantum computer, using as qubits Zeeman sublevels. We emphasize that we take into account only the spontaneous emission of the ions and assume all the other errors have been eliminated.

We calculate the probability to suffer at least one spontaneous emission during the implementation of N quantum gates (Plenio & Knight 1996, 1997). This probability has to be smaller than unity. We represent the qubit by two Zeeman sublevels and use Raman pulses to transfer population between the two states. For the time required to perform N quantum gates, we find $T = N 8\pi \Delta_2/\Omega_{02}^2$, and for the probability for a spontaneous emission from level 2, we find $p_2 = 8\Gamma_{22}N/\Delta_2$. We have to take into account the fact that the two-level approximation can break down. This leads to an additional independent source of spontaneous emission via extraneous levels. One finally obtains the probability to have a spontaneous emission from an extraneous level

$$p_3 = \frac{80\Gamma_{33}^2\pi^2N^2L}{\Delta_{13}^2\beta\eta^2}\left(\frac{\omega_{12}}{\omega_{13}}\right)^3. \tag{4.1}$$

The total probability of a spontaneous emission is $p_{\text{tot}} = p_2 + p_3$ and therefore the error rate per quantum gate is

$$r = \frac{p_{\text{tot}}}{N} = \sqrt{\frac{320L}{\beta}}\frac{\pi\Gamma_{33}}{\Delta_{13}\eta}\left(\frac{\omega_{12}}{\omega_{13}}\right)^{3/2}. \tag{4.2}$$

We use the data for the ions given in Plenio & Knight (1997). If we assume $\eta = 1$, $\beta = 1$ as well as (Knill *et al.* 1996) $L = 7$ and an optimistic $N = 10^6$, we see that even for barium the probability for at least one emission is almost one. The explicit values are for barium $r = 0.44 \times 10^{-6}$, for mercury $r = 9.26 \times 10^{-6}$ and for calcium $r = 2.03 \times 10^{-6}$. This means that unless the encoding procedures given in Knill *et al.* (1996) and Aharonov & Ben-Or (1996) can be improved substantially, the accuracy threshold for quantum computation will not be achievable. Some progress in this direction has been made recently (Steane 1997).

5. Conclusions

We have studied the impact of spontaneous emission on the practical applicability of quantum computation in linear ion traps and especially the possibility of using a quantum computer to factorize large numbers. We conclude that with present technology such a factorization will not be possible even if we employ sophisticated methods of quantum error correction. We then investigated the minimal error rate per quantum gate and compared it to recently established accuracy thresholds that would, in principle, allow arbitrarily complicated quantum computations. We find that the presently known thresholds cannot be achieved because of spontaneous emission alone. Other sources of error would lead to even stronger limitations. We conclude that new physical ideas are therefore necessary if the goal of practically useful quantum computation is to be reached.

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